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Cardinal invariants about shrinkability of unbounded sets

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Abstract

In our previous paper (Eda et al., to appear), we introduced a cardinal invariant $\mathfrak{b}^*$ and studied some properties of the cardinal $\mathfrak{b}^*$. In the present paper we define new cardinal invariants which are related to Cichoń's diagram and generalize the notion of $\mathfrak{b}^*$. We investigate the relations between them and other cardinals which appear in Cichoń's diagram.

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1. Introduction

In our previous paper [5], we introduced a cardinal invariant $\mathfrak{b}^*$, which is defined with the notion of unbounded family in ω^ω but differs from $\mathfrak{b}$. It was a problem in set-theoretic topology that motivated us to consider such a new cardinal invariant.

Let κ be an infinite cardinal. The *sequential fan* S_κ is the following topological space: $S_\kappa = \{\infty\} \cup (\kappa \times \omega)$ as a set, every point of $\kappa \times \omega$ is isolated, and a basic neighborhood of ∞ is of the form $U_\varphi = \{\infty\} \cup \{\langle\alpha, n\rangle\colon n \geqslant \varphi(\alpha)\}$ where $\varphi \in \omega^\kappa$.

For a topological space X, the *tightness* of X, $t(X)$, is the smallest cardinal λ such that for every point $x \in X$ and $A \subseteq X$, if $x \in \operatorname{cl} A$ then there exists $B \subseteq A$ with $|B| \leqslant \lambda$ and $x \in \operatorname{cl} B$.

It is easily seen that $t(S_\kappa) = \omega$ for each κ. But the tightness of the product space of two sequential fans, $t(S_\omega \times S_\kappa)$, is more complicated. In the paper [5] we define a

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new cardinal invariant $\mathfrak{b}^*$ and showed that it gives a combinatorial characterization of the tightness of $S_\omega \times S_\kappa$.

For $f, g \in \omega^\omega$, $f \leqslant^* g$ if $f(n) \leqslant g(n)$ for all but finitely many $n \in \omega$. $\mathfrak{b}$ is the smallest size of the unbounded family of ω^ω with respect to $\leqslant^*$, and $\mathfrak{d}$ is the smallest size of the dominating family of ω^ω with respect to $\leqslant^*$. Clearly $\mathfrak{b} \leqslant \mathfrak{d}$ holds.

We define the cardinal $\mathfrak{b}^*$ and state our results proved in [5].

Definition 1.1. $\mathfrak{b}^*$ is the smallest cardinal λ such that, for every unbounded family $\mathcal{F} \subseteq \omega^\omega$, there exists a subfamily $\mathcal{G} \subseteq \mathcal{F}$ such that $|\mathcal{G}| \leqslant \lambda$ and $\mathcal{G}$ is still unbounded.

Theorem 1.2 [5, Theorem 1.3].
(1) *For* $\omega \leqslant \kappa < \mathfrak{b}$, $t(S_\omega \times S_\kappa) = \omega$ *holds.*
(2) $t(S_\omega \times S_\mathfrak{b}) = \mathfrak{b}$.
(3) *For* $\kappa \geqslant \mathfrak{b}^*$, $t(S_\omega \times S_\kappa) = \mathfrak{b}^*$ *holds.*
In particular, $t(S_\omega \times S_{2^\omega}) = \mathfrak{b}^*$.

Theorem 1.3 [5, Theorem 1.4].
(1) $\mathfrak{b} \leqslant \mathfrak{b}^* \leqslant \mathfrak{d}$.
(2) *Both* $\mathfrak{b} < \mathfrak{b}^*$ *and* $\mathfrak{b}^* < \mathfrak{d}$ *are consistent with* ZFC.

Now we are interested in the cardinal $\mathfrak{b}^*$ itself rather than sequential fans. In the present paper we introduce some cardinal invariants which generalize the notion of $\mathfrak{b}^*$ and investigate their properties.

Let $\mathcal{I}$ denote a σ-ideal on ω^ω containing all singletons. The cardinal $\mathsf{non}(\mathcal{I})$ is the smallest size of a subset of ω^ω which is not in $\mathcal{I}$, and the cardinal $\mathsf{cof}(\mathcal{I})$ is the smallest size of a basis of the ideal $\mathcal{I}$, that is, the smallest size of a subfamily $\mathcal{A}$ of $\mathcal{I}$ such that for every $B \in \mathcal{I}$ there exists $A \in \mathcal{A}$ with $B \subseteq A$. It is easily seen that $\mathsf{non}(\mathcal{I}) \leqslant \mathsf{cof}(\mathcal{I})$ holds for each ideal $\mathcal{I}$.

Analogously to Definition 1.1, we define the following cardinal for the ideal $\mathcal{I}$.

Definition 1.4. $\mathsf{shr}(\mathcal{I})$ is the smallest cardinal λ such that, for every set $A \subseteq \omega^\omega$ with $A \notin \mathcal{I}$, there exists a subset $B \subseteq A$ such that $|B| \leqslant \lambda$ and $B \notin \mathcal{I}$.

Let $\mathcal{N}$, $\mathcal{M}$, $\mathcal{K}$ denote the ideals generated by null sets, meager sets, and σ-compact sets in the Baire space ω^ω respectively. Note that $\mathcal{K}$ is a subideal of $\mathcal{M}$, and it is not so hard to see that $\mathfrak{b} = \mathsf{non}(\mathcal{K})$, $\mathfrak{d} = \mathsf{cof}(\mathcal{K})$ and $\mathfrak{b}^* = \mathsf{shr}(\mathcal{K})$.

For the cardinals $\mathsf{non}(\mathcal{M})$, $\mathsf{non}(\mathcal{N})$, $\mathsf{cof}(\mathcal{M})$, $\mathsf{cof}(\mathcal{N})$, $\mathfrak{b}$ and $\mathfrak{d}$, the following inequalities are known. These inequalities form a part of *Cichoń's diagram.* (See [3,6] for more information.)

Theorem 1.5.
(1) $\mathfrak{b} \leqslant \mathsf{non}(\mathcal{M})$.
(2) (Miller [12, Theorem 3]) $\mathfrak{d} \leqslant \mathsf{cof}(\mathcal{M})$.
(3) (Bartoszyński [1]) $\mathsf{cof}(\mathcal{M}) \leqslant \mathsf{cof}(\mathcal{N})$.

We shall prove the following relations, which generalize Theorem 1.3.

Theorem 1.6.
(1) $\mathsf{non}(\mathcal{M}) \leqslant \mathsf{shr}(\mathcal{M}) \leqslant \mathsf{cof}(\mathcal{M})$, $\mathsf{non}(\mathcal{N}) \leqslant \mathsf{shr}(\mathcal{N}) \leqslant \mathsf{cof}(\mathcal{N})$.
(2) *Each of the following sentences is consistent with* ZFC: $\mathsf{non}(\mathcal{M}) < \mathsf{shr}(\mathcal{M})$, $\mathsf{shr}(\mathcal{M}) < \mathsf{cof}(\mathcal{M})$, $\mathsf{non}(\mathcal{N}) < \mathsf{shr}(\mathcal{N})$, $\mathsf{shr}(\mathcal{N}) < \mathsf{cof}(\mathcal{N})$.

Our notation is standard and we refer the reader to [2] or [10] for undefined notions.

2. Consistency results

In this section we show that the cardinals $\mathsf{shr}(\mathcal{M})$ and $\mathsf{shr}(\mathcal{N})$ are nontrivial, that is, they are consistently different from the cardinals which appear in Cichoń's diagram.

Theorem 2.1. *For any ideal $\mathcal{I}$ on ω^ω, $\mathsf{non}(\mathcal{I}) \leqslant \mathsf{shr}(\mathcal{I}) \leqslant \mathsf{cof}(\mathcal{I})$ holds.*

Proof. $\mathsf{non}(\mathcal{I}) \leqslant \mathsf{shr}(\mathcal{I})$ follows immediately from the definition of $\mathsf{shr}(\mathcal{I})$. To show $\mathsf{shr}(\mathcal{I}) \leqslant \mathsf{cof}(\mathcal{I})$, fix $X \subseteq \omega^\omega$ with $X \notin \mathcal{I}$ and let $\mathcal{A} = \{A_\xi\colon \xi < \mathsf{cof}(\mathcal{I})\}$ a basis of the ideal $\mathcal{I}$. For each $\xi < \mathsf{cof}(\mathcal{I})$ we can find $y_\xi \in X \setminus A_\xi$. Let $Y = \{y_\xi\colon \xi < \mathsf{cof}(\mathcal{I})\} \subseteq X$. Then $|Y| \leqslant \mathsf{cof}(\mathcal{I})$ and $Y \notin \mathcal{I}$. □

Theorem 1.6(1) follows immediately from the above theorem.

Now we turn to the consistency proofs. All required models are obtained by Cohen or random forcing extensions. Through this section M is a c.t.m. for ZFC, $\mathbb{P}$ is a forcing notion satisfying c.c.c., and G is a $\mathbb{P}$-generic filter over M.

Note that every Borel subset of ω^ω can be coded by a function in ω^ω, which is called a *Borel code*. (See [8,11] for details.) For a Borel code c, let $\widehat{c}$ denote the Borel set coded by c.

It is known that the ideals $\mathcal{M}$, $\mathcal{N}$ and $\mathcal{K}$ are generated by Borel sets and absolute in the sense of Borel codes, that is, for every Borel code c the sentence "$\widehat{c}$ is meager" (or "$\widehat{c}$ is measure zero" or "$\widehat{c}$ is σ-compact" respectively) is absolute for all transitive models of ZFC containing c as a Borel code [11].

Definition 2.2. Let $\mathcal{I}$ denote a σ-ideal on ω^ω which is absolute in the sense of Borel codes. We say $\mathbb{P}$ *preserves $\mathcal{I}$-positive sets* if for any $A \subseteq \omega^\omega$ with $A \notin \mathcal{I}$ we have $\Vdash_{\mathbb{P}} A \notin \mathcal{I}$.

Needless to say, "$\mathcal{M}$-positive" means "nonmeager", "$\mathcal{N}$-positive" means "outer measure positive", and "$\mathcal{K}$-positive" means "unbounded".

For an infinite set I, let $\mathbb{C}(I)$, $\mathbb{B}(I)$ denote the Cohen and random forcing notions with the index set I respectively (see [2,10,11]).

Lemma 2.3 [4, Corollary 3.5]. *For any infinite set I, $\mathbb{C}(I)$ preserves unbounded sets.*

Lemma 2.4 [2, Subsections 6.2.2 and 6.2.3]. *For any infinite set I, $\mathbb{C}(I)$ preserves non-meager sets, and $\mathbb{B}(I)$ preserves sets having outer measure positive.*

Definition 2.5. For a forcing notion $\mathbb{P}$, a standard $\mathbb{P}$-name $\dot{f}$ for a real is a name uniquely determined by a system $\{A_{mn}\colon m, n < \omega\}$ with the following:

(1) $A_{mn} \subseteq \mathbb{P}$ is an antichain of $\mathbb{P}$ and $n \neq n'$ implies $A_{mn} \cap A_{mn'} = \emptyset$,

(2) $\bigcup_{n<\omega} A_{mn}$ is a maximal antichain of $\mathbb{P}$, and

(3) For each $p \in A_{mn}$, $p \Vdash_{\mathbb{P}} \dot{f}(m) = n$.

Theorem 2.6. *Assume $2^\omega = \lambda$ and κ be an infinite cardinal. Then,*

(1) *for $\mathbb{P} = \mathbb{C}(\kappa)$ and $\mathcal{I} = \mathcal{K}$ or $\mathcal{I} = \mathcal{M}$, or*

(2) *for $\mathbb{P} = \mathbb{B}(\kappa)$ and $\mathcal{I} = \mathcal{N}$,*

the following holds: any subset A of ω^ω with $A \notin \mathcal{I}$ has a subset $B \subseteq A$ such that $|B| \leqslant \lambda$ and $B \notin \mathcal{I}$ in the forcing extension by $\mathbb{P}$.

Proof. We prove both in the cases (1) and (2) in parallel. Let $\mathbb{P}(I)$ denote $\mathbb{C}(I)$ when $\mathbb{P} = \mathbb{C}(\kappa)$ and $\mathbb{B}(I)$ when $\mathbb{P} = \mathbb{B}(\kappa)$. Note that for every condition $p \in \mathbb{P}$ we can find a countable $J \subseteq \kappa$ and $p' \in \mathbb{P}(J)$ so that $p = p' \times 1_{\mathbb{P}(\kappa\setminus J)}$. We denote such a countable set J by $\mathrm{supp}(p)$.

For an infinite $I \subseteq \kappa$, let $X(I)$ be the collection of all standard $\mathbb{P}(I)$-names for reals and let $\mathcal{X} = X(\kappa)$. It suffices to deal with the case $\kappa > \lambda$.

Suppose the theorem is false, then there are $p_0 \in \mathbb{P}(\kappa)$ and a collection $\dot{A}$ of standard $\mathbb{P}(\kappa)$-names for reals such that

$$p_0 \Vdash_{\mathbb{P}(\kappa)} \text{“}\dot{A} \notin \mathcal{I} \wedge \forall B \subseteq \dot{A}\,(|B| \leqslant \lambda \to B \in \mathcal{I})\text{”}.$$

Let $S = \{X(I)\colon I \in [\kappa]^\lambda \wedge \mathrm{supp}(p_0) \subseteq I\}$, then $S \subseteq [\mathcal{X}]^\lambda$. S is stationary, since it is unbounded and closed under unions of increasing ω_1-sequences. By assumption and the fact that $\mathbb{P}(I)$ preserves $\mathcal{I}$-positive sets for any infinite set I, for each $X = X(I) \in S$ we get a standard $\mathbb{P}(I)$-name $\dot{c}_X$ for a Borel code so that

$$p_0 \Vdash_{\mathbb{P}(I)} \text{“}\dot{A} \cap X \subseteq \widehat{\dot{c}_X} \wedge \widehat{\dot{c}_X} \in \mathcal{I}\text{”}.$$

Note that $\dot{c}_X \in X = X(I)$. By Fodor's lemma for $[\mathcal{X}]^\lambda$ (see [7, Theorem 3.2]) there is a stationary set $S' \subseteq S$ and a standard $\mathbb{P}(\kappa)$-name $\dot{c}$ for a Borel code such that $\dot{c}_X = \dot{c}$ for all $X \in S'$. Clearly $p_0 \Vdash_{\mathbb{P}(\kappa)}$ “$\widehat{\dot{c}} \in \mathcal{I}$”. Since S' is unbounded in $[\mathcal{X}]^\lambda$, we have $p_0 \Vdash_{\mathbb{P}(\kappa)}$ “$\dot{x} \in \widehat{\dot{c}}$” for all $\dot{x} \in \dot{A}$. It means that $p_0 \Vdash_{\mathbb{P}(\kappa)}$ “$\dot{A} \subseteq \widehat{\dot{c}}$” and hence $p_0 \Vdash_{\mathbb{P}(\kappa)}$ “$\dot{A} \in \mathcal{I}$”, which is a contradiction. □

Corollary 2.7. *Assume* CH *and let κ be a cardinal of uncountable cofinality. Then,*

(1) $\mathfrak{b} = \mathfrak{b}^* = \mathsf{non}(\mathcal{M}) = \mathsf{shr}(\mathcal{M}) = \omega_1$ *and* $\mathfrak{d} = \mathsf{cof}(\mathcal{M}) = \kappa$ *hold in the forcing model by* $\mathbb{C}(\kappa)$.

(2) $\mathsf{non}(\mathcal{N}) = \mathsf{shr}(\mathcal{N}) = \omega_1$ *and* $\mathsf{cof}(\mathcal{N}) = \kappa$ *hold in the forcing model by* $\mathbb{B}(\kappa)$.

Using Lemmas 2.3, 2.4 and Theorem 2.6, we can easily prove all the remaining consistency results.

Lemma 2.8. *Let $\mathcal{I}$ denote a σ-ideal on ω^ω which is absolute in the sense of Borel codes. If $\mathbb{P}$ preserves $\mathcal{I}$-positive sets, then $(\mathsf{shr}(\mathcal{I}))^M \leqslant (\mathsf{shr}(\mathcal{I}))^{M[G]}$ holds.*

Proof. Fix $\lambda < \mathsf{shr}(\mathcal{I})$ arbitrarily. Let $A \subseteq \omega^\omega \cap M$ be the set such that $A \notin \mathcal{I}$ and for $B \subseteq A$ if $|B| \leqslant \lambda$ then $B \in \mathcal{I}$. We show that A witnesses $(\mathsf{shr}(\mathcal{I}))^{M[G]} > \lambda$.

Since $\mathbb{P}$ preserves $\mathcal{I}$-positive sets, $\Vdash_{\mathbb{P}}$ "$A \notin \mathcal{I}$" also holds. Fix a $\mathbb{P}$-name $\dot{B}$ such that $\Vdash_{\mathbb{P}}$ "$\dot{B} \subseteq A \wedge |\dot{B}| \leqslant \lambda$". Since $\mathbb{P}$ is c.c.c., we can find $B' \subseteq A$ such that $|B'| \leqslant \lambda$ and $\Vdash_{\mathbb{P}}$ "$\dot{B} \subseteq B'$". By assumption, we can find a Borel code c so that $B' \subseteq \widehat{c} \in \mathcal{I}$ holds. By the absoluteness of $\mathcal{I}$,

$$\Vdash_{\mathbb{P}} \text{“}\dot{B} \subseteq B' \subseteq (\widehat{c})^M = \widehat{c} \cap M \subseteq \widehat{c} \in \mathcal{I}\text{”},$$

so $\Vdash_{\mathbb{P}}$ "$\dot{B} \in \mathcal{I}$" holds. □

Corollary 2.9. *Assume* MA + $\omega_1 < 2^\omega = \lambda \leqslant \kappa$ *and κ has uncountable cofinality. Then,*

(1) $\mathfrak{b} = \mathsf{non}(\mathcal{M}) = \omega_1$, $\mathfrak{b}^* = \mathsf{shr}(\mathcal{M}) = \lambda$ *and* $\mathfrak{d} = \mathsf{cof}(\mathcal{M}) = \kappa$ *hold in the forcing model by $\mathbb{C}(\kappa)$.*

(2) $\mathsf{non}(\mathcal{N}) = \omega_1$, $\mathsf{shr}(\mathcal{N}) = \lambda$ *and* $\mathsf{cof}(\mathcal{N}) = \kappa$ *hold in the forcing model by $\mathbb{B}(\kappa)$.*

3. Adding a Cohen real can change $\mathfrak{b}^*$

We notice the fact that $(\mathfrak{b}^*)^M = (\mathfrak{b}^*)^{M[G]}$ holds in all the forcing models obtained in Corollaries 2.7 and 2.9. So it is natural to ask whether $\mathfrak{b}^*$ is absolute for Cohen or random forcing extensions. In fact, the answer is 'no' for Cohen forcing. This section is devoted to show that adding a Cohen real can change $\mathfrak{b}^*$.

Here we consider meager or nonmeager sets in the Cantor space 2^ω instead of the Baire space ω^ω. The cardinals associated with the meager ideal are the same if we used either of two spaces.

For $x, y \in 2^\omega$, $x + y$ is defined as $(x + y)(n) = x(n) + y(n) \bmod 2$ for each $n \in \omega$. Similarly, for $s, t \in 2^{<\omega}$, $s + t$ is defined as $\mathrm{dom}\,(s + t) = \min\{|s|, |t|\}$ and $(s + t)(n) = s(n) + t(n) \bmod 2$ for $n \in \mathrm{dom}\,(s + t)$. For $A \subseteq 2^\omega$ and $z \in 2^\omega$ we define $A + z = \{x + z\colon x \in A\}$.

For $r \in [\omega]^\omega$ the increasing enumeration of r is denoted by f_r. Then we will say $X \subseteq [\omega]^\omega$ is *bounded* if the family of functions $\{f_r\colon r \in X\}$ is bounded with respect to $\leqslant^*$, and otherwise we say it is *unbounded*. In the following argument we will identify an infinite subset of ω with its characteristic function. It is easy to see that every nonmeager set in 2^ω is unbounded.

Theorem 3.1. *Let κ be a cardinal and $X \in M$ be a nonmeager subset of 2^ω such that, for $Y \in M$ if $Y \subseteq X$ and $|Y| < \kappa$ then Y is meager. Let c be a Cohen real over M. Then in $M[c]$,*

(1) *$X + c$ is unbounded and*

(2) *Every $Z \subseteq X + c$ of size less than κ is bounded.*

Because any element of $X + c$ takes infinitely often the value 1, (1) and (2) make sense.

Corollary 3.2. $(\mathsf{shr}(\mathcal{M}))^M \leqslant (\mathfrak{b}^*)^{M[c]}$. *In particular, if* $(\mathfrak{b}^*)^M < (\mathsf{shr}(\mathcal{M}))^M$, *then* $(\mathfrak{b}^*)^M < (\mathfrak{b}^*)^{M[c]}$.

Of course, $\mathfrak{b}^* < \mathsf{shr}(\mathcal{M})$ is consistent with ZFC. (Consider the model obtained by adding ω_2 random reals to the model of CH.)

A subtree T of $2^{<\omega}$ with no maximal node is called *nowhere dense* if for any $t \in 2^{<\omega}$ there exists $s \in 2^{<\omega} \setminus T$ such that $t \subseteq s$. Note that for any $X \subseteq 2^\omega$ the subtree $T(X) = \{r \restriction n\colon r \in X \wedge n \in \omega\}$ of $2^{<\omega}$ is nowhere dense if and only if X is nowhere dense in 2^ω.

Lemma 3.3. *Let T be a nowhere dense tree and $k \in \omega$. Then for any $p \in 2^{<\omega}$ there exists $q \in 2^{<\omega}$ with $p \subseteq q$ such that:*

$$\forall t \in \mathrm{Lev}_{|q|}(T)\ \bigl(k < \bigl|(t+q)^{-1}\{1\}\bigr|\bigr)$$

Proof. Fix a nowhere dense tree T and $p \in 2^{<\omega}$ arbitrarily. For $q \in 2^{<\omega}$ with $p \subseteq q$ and $t \in \mathrm{Lev}_{|q|}(T)$ we say t *survives* q if

$$\bigl|(t+p)^{-1}\{1\}\bigr| = \bigl|(t+q)^{-1}\{1\}\bigr|.$$

In order to prove the lemma it suffices to show that there exists $q \in 2^{<\omega}$ with $p \subseteq q$ such that no node in $\mathrm{Lev}_{|q|}(T)$ survives q. So we will inductively construct a finite sequence $p = p_0 \subset p_1 \subset \cdots \subset p_l$ such that:

- $|\{t \in \mathrm{Lev}_{|p_i|}(T)\colon t \text{ survives } p_i\}| > |\{t \in \mathrm{Lev}_{|p_{i+1}|}(T)\colon t \text{ survives } p_{i+1}\}|$ for $i < l$ and
- $|\{t \in \mathrm{Lev}_{|p_l|}(T)\colon t \text{ survives } p_l\}| = 0$,

and let $q = p_l$. Assume that p_i is constructed. If there is no node $t \in \mathrm{Lev}_{|p_i|}(T)$ surviving p_i, let $l = i$ and we are done. Otherwise, pick a node $t \in \mathrm{Lev}_{|p_i|}(T)$ surviving p_i. Since T is nowhere dense, there exists $s \in T$ such that $t \subset s$ and $s \in 2^{<\omega} \setminus T$. We define $p_{i+1} \in 2^{|s|}$ as follows.

$$p_{i+1}(n) = \begin{cases} p_i(n) & (n < |p_i|), \\ s(n) & (|p_i| \leqslant n < |s|). \end{cases}$$

Since $t \in \mathrm{Lev}_{|p_{i+1}|}(T)$ survives p_{i+1} if and only if t and p_{i+1} coincide on $|p_{i+1}| \setminus |p_i|$ and $t \restriction |p_i|$ survives p_i, it is clear that this p_{i+1} satisfies the condition. $\square$

Proof of Theorem 3.1. (1) It is clear from the fact mentioned above, because X is still nonmeager in $M[c]$ by Lemma 2.3 and hence so is $X + c$.

(2) There is $Y \in M$ of the same size as Z such that $Y \subseteq X$ and $Z \subseteq Y + c$. Then we will work in M. Since Y is meager, there is a countable family $\{Y_m\colon m \in \omega\}$ of nowhere dense sets which covers Y. Let T_m be the nowhere dense tree $T(Y_m)$. Fix $m \in \omega$ and we will construct a $\mathbb{C}$-name $\dot{g}_m$ such that for any infinite path r through T_m we have

$\Vdash \forall n < \omega\, (f_{r+c}(n) \leqslant \dot{g}_m(n))$. We will use $2^{<\omega}$ with inverse inclusion $\supseteq$ as our Cohen forcing notion $\mathbb{C}$ instead of $\mathrm{Fn}(\omega, 2)$. Using Lemma 3.3 we get a maximal antichain A_k for each $k \in \omega$ so that $k < |(t+q)^{-1}\{1\}|$ for any $q \in A_k$ and any $t \in \mathrm{Lev}_{|q|}(T)$. We define a $\mathbb{C}$-name $\dot{g}_m$ as

$$\Vdash \text{“}\dot{g}_m(k) = |q| \text{ iff } \dot{G} \cap A_k = \{q\}\text{”}$$

for any $k \in \omega$. Clearly this $\dot{g}_m$ is what we need. At last we take a $\mathbb{C}$-name $\dot{g}$ so that $\Vdash_{\mathbb{C}}$ “$\forall m \in \omega\, (\dot{g}_m \leqslant^* \dot{g})$”, then it witnesses the boundedness of Z. $\square$

4. Cofinality of $\mathfrak{b}^*$

It is well known that $\mathfrak{b}$ is regular and $\mathfrak{d}$ has uncountable cofinality. (More precisely, $\mathrm{cf}(\mathfrak{d}) \geqslant \mathfrak{b}$.) Now we observe that $\mathfrak{b}^*$ also has uncountable cofinality.

Definition 4.1. For $A, B \subseteq \omega$, $A \subseteq^* B$ if $A \setminus B$ is finite. A family $\{B_\alpha\colon \alpha < \lambda\}$ of infinite subsets of ω is called a *tower* if $B_\beta \subseteq^* B_\alpha$ for $\alpha < \beta < \lambda$ and there is no infinite $B \subseteq \omega$ satisfying $B \subseteq^* B_\alpha$ for all $\alpha < \lambda$.

Definition 4.2. $\mathfrak{t}$ is the smallest size of a tower.

It is known that $\mathfrak{t}$ is regular and $\omega_1 \leqslant \mathfrak{t} \leqslant \mathfrak{b}$ (see [13, Theorem 3.1]). The following theorem shows that $\mathfrak{b}^*$ has uncountable cofinality.

Theorem 4.3. $\mathrm{cf}(\mathfrak{b}^*) \geqslant \mathfrak{t}$.

Proof. Assume that $\mathrm{cf}(\mathfrak{b}^*) = \lambda < \mathfrak{t}$. Then there are κ_α, $\mathcal{F}_\alpha$ for $\alpha < \lambda$ such that

(1) $\kappa_\alpha < \kappa_\beta < \mathfrak{b}^*$ for $\alpha < \beta < \lambda$ and $\sup\{\kappa_\alpha\colon \alpha < \lambda\} = \mathfrak{b}^*$,

(2) $\mathcal{F}_\alpha$ is an unbounded family of non-decreasing functions in ω^ω and $|\mathcal{F}_\alpha| = \kappa_\alpha$,

(3) For any $\mathcal{F} \subseteq \mathcal{F}_\alpha$ if $|\mathcal{F}| < \kappa_\alpha$ then $\mathcal{F}$ is bounded.

For each $\alpha < \lambda$, let $\mathcal{S}_\alpha = [\mathcal{F}_\alpha]^{<\kappa_\alpha}$. Set

$$\mathcal{S} = \left\{S \in \prod_{\alpha<\lambda} \mathcal{S}_\alpha\colon \exists\beta < \lambda\, \forall\alpha < \lambda\, \left(\left|S(\alpha)\right| \leqslant \kappa_\beta\right)\right\}.$$

Then for any $S \in \mathcal{S}$, $\bigcup_{\alpha<\lambda} S(\alpha)$ is bounded. So we can take $h_S \in \omega^\omega$ for each $S \in \mathcal{S}$ such that $f \leqslant^* h_S$ for all $f \in \bigcup_{\alpha<\lambda} S(\alpha)$.

Define a quasi-order $\prec$ on $\mathcal{S}$ by $S \prec T$ iff there exists $\beta < \lambda$ such that $S(\alpha) \subseteq T(\alpha)$ for all α with $\beta \leqslant \alpha < \lambda$. For each $S \in \mathcal{S}$, let $\mathcal{H}_S = \{f \in \omega^\omega\colon \forall T \succ S\, (f \leqslant^* h_T)\}$, and let $\mathcal{H} = \bigcup_{S\in\mathcal{S}} \mathcal{H}_S$. We shall show that $\mathcal{H}$ is unbounded and that any subfamily $\mathcal{G} \subseteq \mathcal{H}$ with $|\mathcal{G}| < \mathfrak{b}^*$ is bounded, which contradicts $\mathrm{cf}(\mathfrak{b}^*) < \mathfrak{t} \leqslant \mathfrak{b}$.

First we show that $\mathcal{H}$ is unbounded. So take $h \in \omega^\omega$ arbitrarily. By the assumption that $\lambda < \mathfrak{t}$, we can take $f_\alpha \in \mathcal{F}_\alpha$ and infinite $B_\alpha \subseteq \omega$ by induction on $\alpha < \lambda$ satisfying the following: $B_\alpha = \{k < \omega\colon h(k) < f_\alpha(k)\}$ for each $\alpha < \lambda$, and $B_\beta \subseteq^* B_\alpha$ for $\alpha < \beta < \lambda$. Choose infinite $B \subseteq \omega$ so that $B \subseteq^* B_\alpha$ for all $\alpha < \lambda$.

Let $S = \langle\{f_\alpha\}\colon \alpha < \lambda\rangle$, and define $f \in \omega^\omega$ by $f(k) = h(k) + 1$ for $k \in B$ and $f(k) = 0$ otherwise. Then $f \not\leqslant^* h$. On the other hand, $f \leqslant f_\alpha$ for all $\alpha < \lambda$, and if $T \succ S$ then there is $\alpha < \lambda$ such that $f_\alpha \leqslant^* h_T$, so $f \in \mathcal{H}_S$ and hence $f \in \mathcal{H}$.

Next we show that $\mathcal{G}$ is bounded for all $\mathcal{G} \subseteq \mathcal{H}$ with $|\mathcal{G}| < \mathfrak{b}^*$. Fix such $\mathcal{G} \subseteq \mathcal{H}$ and take $\beta_0 < \lambda$ so that $|\mathcal{G}| < \kappa_{\beta_0}$. For each $g \in \mathcal{G}$, take $S_g \in \mathcal{S}$ and $\beta_g < \lambda$ such that $g \in \mathcal{H}_{S_g}$ and $|S_g(\alpha)| \leqslant \kappa_{\beta_g}$ for all $\alpha < \lambda$. For each $\alpha \in (\beta_0, \lambda)$, let $\mathcal{G}_\alpha = \{g \in \mathcal{G}\colon \beta_g < \alpha\}$. We claim that $\mathcal{G}_\alpha$ is bounded for all $\alpha \in (\beta_0, \lambda)$, which implies that $\mathcal{G} = \bigcup_{\beta_0<\alpha<\lambda} \mathcal{G}_\alpha$ is bounded. So fix $\alpha \in (\beta_0, \lambda)$. Since $\bigcup_{g\in\mathcal{G}_\alpha} S_g(\xi) \in \mathcal{S}_\xi$ for all $\xi \in (\alpha, \lambda)$, we can define $T \in \mathcal{S}$ by $T(\xi) = \bigcup_{g\in\mathcal{G}_\alpha} S_g(\xi)$ if $\alpha \leqslant \xi < \lambda$ and $T(\xi) = \emptyset$ otherwise. Since $g \in S_g$ and $S_g \prec T$ for all $g \in \mathcal{G}_\alpha$, we have $g \leqslant^* h_T$ for all $g \in \mathcal{G}_\alpha$. □

5. Questions

As we mentioned in Section 1, $\mathfrak{b} \leqslant \mathsf{non}(\mathcal{M})$ and $\mathfrak{d} \leqslant \mathsf{cof}(\mathcal{M})$ hold in ZFC. So it is natural to ask the following question.

Question 1. *Does* $\mathfrak{b}^* \leqslant \mathsf{shr}(\mathcal{M})$ *hold in* ZFC?

In Section 3 we proved that adding a Cohen real can change $\mathfrak{b}^*$.

Question 2. *Can adding random reals change* $\mathfrak{b}^*$?

Note that the dual result to Corollary 3.2 cannot hold, because adding random reals cannot raise $\mathfrak{d}$, and $\mathfrak{d} < \mathsf{non}(\mathcal{N})$ is consistent with ZFC (see [3]).

Question 3. *Is* $(\mathsf{shr}(\mathcal{M}))^M = (\mathfrak{b}^*)^{M[G]}$ *for Cohen forcing*?

Question 4. *What happens about* $\mathsf{shr}(\mathcal{M})$ *or* $\mathsf{shr}(\mathcal{N})$ *when Cohen or random reals are added*?

In Section 4 we proved that $\mathrm{cf}(\mathfrak{b}^*) \geqslant \mathfrak{t}$. But it is known that $\mathfrak{t} < \mathfrak{b}$ is consistent with ZFC (see [13, Theorem 5.3]).

Question 5. *Does* $\mathrm{cf}(\mathfrak{b}^*) \geqslant \mathfrak{b}$ *hold in* ZFC?

Added in proof. After the submission of this paper, the second author proved that $\mathfrak{b}^* \leqslant \mathsf{shr}(\mathcal{M})$, which gives the answer to Question 1. On the other hand, the first author proved that $\mathsf{shr}(\mathcal{N}) < \mathfrak{b}$ is consistent with ZFC. So we have proved all ZFC-provable inequalities between two cardinals from this paper. We will publish these results in the near future.

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